MOTION OF A STRING VIBRATING AGAINST A RIGID FIXED OBSTACLE

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Abstract. We consider a string, fixed at both ends and moving in a plane in presence of a straight fixed obstacle placed on the equilibrium position of the string; the rebound of the string on the obstacle obeys the law of perfect reflection. The string being initially at rest in an arbitrary shape, we prove that the motion is periodic with the same period that the free oscillations.

INTRODUCTION AND STATEMENT OF THE PROBLEM

During the past few years several works have been devoted to the motion of strings vibrating in the presence of obstacles. Amerio and Prouse are the first to have considered the problem (1), followed by Schatzman (2), Citrini (3), Cabannes (4), (5), Haraux and Cabannes (6). In the present work, we consider the small oscillations of a vibrating string with fixed ends (0,0) and (1,0) in a plane with a normalized coordinate system Oxu. The free oscillations of the string are perturbed by the presence of a fixed straight obstacle on the equilibrium position (u = 0); the string rebounds against the obstacle following the law of perfect reflection.

The function u(x,t) which represents, at time t, the position of the point of abscissa x satisfies the conditions:

$$(1)\quad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{if } u(x,t) > 0,$$

$$(2)\quad \frac{\partial u}{\partial t}(x,t+0) = -\frac{\partial u}{\partial t}(x,t-0) \quad \text{if } u(x,t) = 0$$

$$(3)\quad u(0,t) = u(1,t) = 0 \quad \text{for } t \geq 0,$$

$$(4)\quad u(x,0) = u_0(x) \geq 0 \text{ for } 0 \leq x \leq 1$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \qquad \text{for } 0 \leq x \leq 1.$$

The function $u_0(x)$, nil for x = 0 and x = 1 and strictly positive for $0 < x < 1$, possesses n relative maxima M_i and n - 1 relative minima m_i in $0 < x < 1$. The relative maxima M_i are reached at $x = a_i$ $(0 < a_1 < a_2 < \dots a_n < 1)$ and the relative minima m_i are reached at $x = b_i$ $(a_i < b_i < a_{i+1})$ (see Fig. 1). We denote by $\alpha(x)$ the odd function, periodic with period 2, equal to $u_0(x)$ for $0 \leq x \leq 1$.

If there was no obstacle, the motion of the string would be the free oscillation: u = w, periodic function in time, with period 2:

$$(5)\quad w(x,t) = \frac{\alpha(t+x) - \alpha(t-x)}{2}$$

MOTION IN THE PRESENCE OF THE OBSTACLE u = 0

To determine the motion of the string in the presence of the straight obstacle placed on the equilibrium position u = 0, it is convenient to introduce the new function

$$(6)\quad v(x,t) = \frac{\alpha(t+x) + \alpha(t-x)}{2} = w(t,x);$$

we will show that, depending on the values of x and t, we have $u = |w|$ or $u = -m_i \pm v$. For that purpose we will determine the values of the function u(x,t) in the square $0 \leq x \leq 1$, $0 \leq t \leq 1$: figure 2. We divide this square in 4 regions:

Region (1)	$0 \leq x \pm t \leq 1$	$t \geq 0$
Region (2)	$0 \leq t \pm x \leq 1$	$x \geq 0$
Region (3)	$0 \leq x - t \leq 1 \leq x + t \leq 2$	$x \leq 1$
Region (4)	$0 \leq t - x \leq 1 \leq t + x \leq 2$	$t \leq 1$

Initially the motion of the string is the free oscillation; in the region (1) we have $\alpha(t+x) \geq 0$ because $0 \leq t + x \leq 1$, and $\alpha(t-x) \leq 0$; also in this region $w(x,t) \geq 0$ and $u(x,t) = w(x,t)$.

To determine the motion of the string in the region (2), we associate to each point M(x,t) in this region, a straight line (segment PQ in the figure 1) parallel to the axis of the abscissae, such that $x_P = t + x$, $x_Q = t - x$ and all the points of PQ are inside or on the frontier of the domain (Ω) limited by the axis of abscissae and the curve (Γ) $u = u_0(x)$ but with at least one point on the curve (Γ). At each segment PQ so defined corresponds one point M of the region (2) and conversely. Three cases are possible:

(a) the point Q is on the curve (Γ)
(b) the point P is one the curve (Γ)
(c) the segment PQ meets the curve (Γ) at one of minima m_i

The region (2) is also divided into sub-regions of types (2.a), (2.b) or (2.c); the last are rectangles limited by characteristics and with one of the vertices on the axis x = 0 (for $x_P = x_Q = b_i$). On the frontier between two regions, two of the conditions (a), (b), or (c) are satisfied. We have

$$(7)\quad \begin{cases} 2\,w(x,t) = \alpha(x_P) - \alpha(x_Q) \\ 2\,v(x,t) = \alpha(x_P) + \alpha(x_Q) \end{cases}$$

and we will prove that the values of the function u(x,t) are

$$(8)\quad \begin{cases} u(x,t) = w(x,t) & \text{in the sub-regions (2.a)} \\ u(x,t) = -w(x,t) & \text{in the sub-regions (2.b)} \\ u(x,t) = v(x,t) - m_i & \text{in the sub-regions (2.c)} \end{cases}$$

As the uniqueness of the solution of the problem (1)-(4) has been proved by Amerio and Prouse (1), it is sufficient to verify that the function u(x,t) defined by the formulae (8) satisfies the conditions (1), (2), (3) and is equal to w along the frontier between the regions (1) and (2).

Condition (1) - The function $u(x,t)$ is always positive or zero and is a solution of equation (1) inside the sub-regions.

Condition (3) - When the point M is the end $x = 0$ of the string, $x_p = x_Q$ and $w = 0$; the segment PQ belongs to a frontier of a sub-region (2.a) or of a sub-region (2.b); as a consequence $u = \pm w = 0$.

Condition (2) - When $u = 0$, there are two possibilities: $v = m_i$ or $w = 0$. The first case is possible only if $\alpha(x_p) = \alpha(x_Q) = m_i$, and the second if $\alpha(x_p) = \alpha(x_Q)$. If $x_p = x_Q$, M is the end $x = 0$ of the string and we have seen that the condition (3) is satisifed; if $x_p \neq x_Q$ the point M is on the frontier between a subregion (2.a) and a subregion (2.b), and across this frontier the sign of $(\partial u/\partial t)$ changes because w becomes -w; therefore the condition (2) is satisfied.

The frontier between the regions (1) and (2) corresponds to the case where the point Q is in 0; on this frontier, which therefore belongs to a subregion (2.a), we have $u = w$; and for similar reason $u = -w$ on the frontier between the region (2) and the region (4).

By the same method we can define in the region (3) a function $u(x,t)$ which satisfies the conditions (1), (2) and (3) and which is equal to w on the frontier between the regions (1) and (3) and to -w on the frontier between the regions (3) and (4); in fact we obtain $u(x,t) = u(1 - x, 1 - t)$ and if the point (x,t) is in the region (3) the point $(1 - x, 1 - t)$ is in the region (2).

As $u = w$ is solution of the problem (1)-(4) in the region (1), the function $u(x,t)$ defined by the formulae (8) and by the relation $u(x,t) = u(1 - x, 1 - t)$ is the solution of the same problem in the regions (2) and (3); and in region (4) we have $u = -w \geq 0$.

The formulae (8) can be expressed in a global form:

$$(9)\quad u(x,t) = \frac{\alpha(t+x) + \alpha(t-x)}{2} - \inf_{\lambda \in R} \alpha(\lambda)$$

where R is the interval $t - x \leq \lambda \leq t + x$. The formula (9) can then be extended to all values of x and t. To do that extension we add to $t + x$ and $t - x$ a multiple (positive or negative, or zero) of 2, so that the result is in the interval 0,2; that means that we write

$$(10)\quad \left| \begin{array}{l} R_+ = t + x - 2[\frac{t + x}{2}] \\ R_- = 2 + t - x - 2[\frac{t - x + 2}{2}] \end{array} \right.$$

where $[x]$ is the integer part of x (the greatest integer smaller than x or equal to x). If R is the interval between the values R_+ and R_-, we can write

$$(11)\quad u(x,t) = \frac{|\alpha(t + x)| + |\alpha(t - x)|}{2} - \inf|\alpha(\lambda)|.$$

CONCLUSION

From the former results, we conclude:

(10) $u(x,1) = u(1 -x, 0)$, $(\partial u/\partial t)(x,1) = 0$.
(11) $u(x,2) = u(x,0)$, $(\partial u/\partial t)(x,2) = 0$.

Theorem - A vibrating string fixed at both ends moves in the presence of a straight fixed obstacle on the equilibrium position of the string, and rebounds on the obstacle following the law of the perfect reflection. The string being initially at rest with an arbitrary shape, the motion is a periodic function of the time, with the same period as in the free oscillation.

Remarks.
1/ If the initial shape is symmetric ($u_o(x) = u_o(1-x)$), the half of the period of the free oscillation is also a period.
2/ If the initial shape is unimodal, that is if the function $u_o(x)$ has only one maximum in the interval $0 \leq x \leq 1$, then there are no sub-regions (2.c) and the motion of the string is:

$$(12)\quad u(x,t) = |w(x,t)|$$

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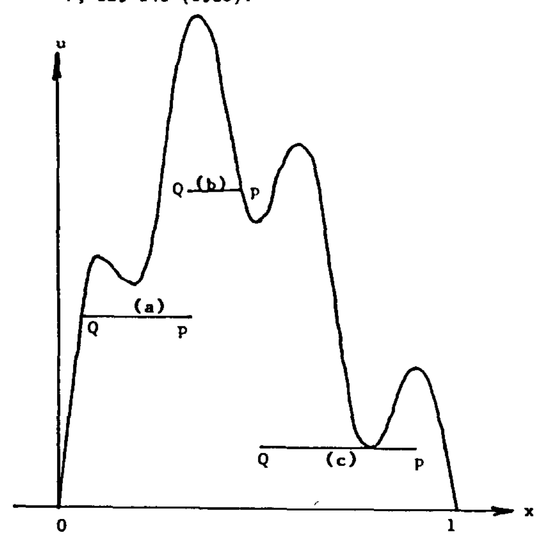

Fig. 1 - Initial shape of the string : $u = u_o(x)$

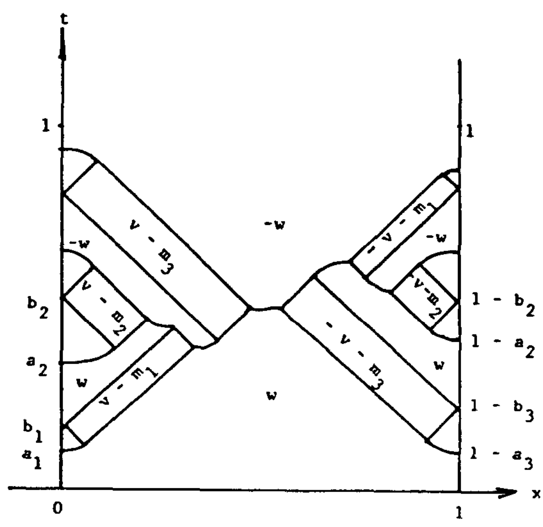

Fig. 2 - Solution in the x,t plane